Low-dimensional chaos in an instance of epilepsy
(chaotic attractors/electroencephalogram/Lyapunov exponents/phase space)

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ABSTRACT Using a time series obtained from the electroencephalogram recording of a human epileptic seizure, we show the existence of a chaotic attractor, the latter being the direct consequence of the deterministic nature of brain activity. This result is compared with other attractors seen in normal human brain dynamics. A sudden jump is observed between the dimensionalities of these brain attractors (4.05 ± 0.05 for deep sleep) and the very low dimensionality of the epileptic state (2.05 ± 0.09). The evaluation of the autocorrelation function and of the largest Lyapunov exponent allows us to sharpen further the main features of underlying dynamics. Possible implications in biological and medical research are briefly discussed.

Recent progress in the theory of nonlinear dynamical systems has provided new methods for the study of time series in such fields as hydrodynamics (1), chemistry (2), climatic variability (3, 4), biochemistry (5, 6), and human brain activity (7). The study of such complex systems may be performed by analyzing experimental data recorded as a series of measurements in time of a pertinent and easily accessible variable of the system. In most cases, such variables describe a global or averaged property of the system. For example, a time series may be obtained by recording at regular time intervals the mean electrical activity of a portion of the mammalian cortex. Although it may seem that such data offer only a one-dimensional view of the activity of the brain, this is not the case: it can be shown that a time series may provide information about a large data recorded during various stages of the sleep cycle. In the final section, we discuss the relevance of the EEG analyses to the understanding of brain activity.

Epileptic Attractor

Epileptic seizures reflect a pathological state of the brain activity, which may occur spontaneously as a result of functional disorders or lesions, or may be induced by various means. There are several forms of epilepsy (9); here we are concerned with seizures of short duration (≈5 sec) known as “petit mal.” This type of generalized epilepsy may invade the entire cerebral cortex and shows a bilateral symmetry between the two hemispheres. During the seizure, the EEG activity suddenly switches into an apparently oscillating mode. A succession of more or less regular and extremely coherent waves of ≈3 cycles per sec may be seen. The waves are separated by less regular spikes. Fig. 1 shows four simultaneous recordings during a seizure. Channels 1 and 2 form the basis of our time series, which will be used for the construction of the phase space trajectories.

Let us assume that the dynamics of the brain activity is described by a set of \{X_0(t), X_1(t), ..., X_{n-1}(t)\} variables satisfying a system of first-order differential equations. A differential equation of order n with a single variable \(X_0\), accessible from experimental data, is equivalent to the original set. Now both \(X_0\) and its derivatives, therefore the ensemble of \(n\) variables, can be obtained from a single time series. However,
FIG. 1. EEG frontal portraits in (a) muscular and also (b) noise-prone channel 2 (left) and channel 3 (right) at 2250 = 450 Hz. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 2. Evolution of the trajectories emanating from the attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 3. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 4. In Fig. 3, we show the unfolding in time for the pseudocycle at an attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 5. Part of the pseudocycle at an attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 6. Part of the pseudocycle at an attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 7. Part of the pseudocycle at an attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)

FIG. 8. Part of the pseudocycle at an attractor as shown in Fig. 1. Topological Aspects of Spatio-Dynamical Systems. Averaging over 24 hr of the data, we can observe that the trajectories covered part of the pseudocycle. They are identical to the one obtained from a phase portrait, which represents the spike activity, which is formed from the one spike activity, which is formed from the pseudocycle. This plane portrait, which is a projection of the wave and activity, was filtered below 0.2 Hz and above 45 Hz and also was used. The signal was obtained from a single patient. Digital equipment was used. The spikes were identified by 3514 Neurobiology: Babloyantz and Destexhe Proc. Natl. Acad. Sci. USA 83 (1986)
more accessible correlation dimension \((11, 12) \nu = D\), which may be obtained easily from numerical analysis of the time series in Fig. 1. Notice that if \(p\) represents the embedding dimension of the attractor then, necessarily, \(D \leq p\).

We introduce a vector notation: \(V_i(t)\) stands for a point of phase space whose coordinates are \(\{V_i(t), V_i(t + \tau), \ldots, V_i(t + (n - 1)\tau)\}\). A "reference" point \(V_i\) from these data is chosen and its distances \(|V_i - V_j|\) from the \(n - 1\) remaining points are computed. This allows us to count the data points that are within a prescribed distance \(r\) from the point \(V_i\) in the phase space. Repeating the process for all values of \(i\), one arrives at the quantity \(C(r)\), which is the integral correlation function of the attractor. The nonvanishing of \(C(r)\) measures the extent to which the presence of a data point \(V_i\) affects the position of the other points. One shows that for small \(r\), \(C(r) \sim r^n\), and the correlation dimension \(\nu\) of the attractor is therefore given by the slope of \(\log C(r)\) versus \(\log r\).

With the help of this last relation, the dimension \(\nu\) is computed by considering successively higher embedding dimensions \(p\) of the phase space. If the \(\nu\) versus \(p\) dependence is saturated beyond some relatively small \(p\), the system represented by the time series should possess an attractor. The saturation value \(\nu_s\) is regarded as the dimensionality of the attractor of the system represented by the time series. The value of \(p\) beyond which saturation is observed provides the minimum number of variables necessary to model the dynamics of the attractor.

The slope of the curve \(\log C(r)\) versus \(\log r\) (Fig. 4) has been evaluated with extreme care. After determining the boundaries of the linear zone by visual inspection, we determine the slope of \(m\) first points in this segment by the least-squares method. The operation is repeated all along the linear region by sliding \(m\) one point further. The computation is repeated for increasing values of \(m\). If the region is linear, all these operations must yield the same value of the slope (within acceptable error boundaries).

Although in principle every value of time lag \(\tau\) is acceptable for the resurrection of the system's dynamics, in practice, for a given time series, only a well-defined range of \(\tau\) (here, \(17 \Delta t \leq \tau \leq 25 \Delta t\)) gives satisfactory linear regions or well-behaved saturation curves.

Fig. 5 shows a saturation curve computed from the epileptic signal. For comparison, the behavior obtained from a random process such as gaussian white noise is drawn. We find a satisfactory saturation beyond the embedding dimension five, which yields a correlation dimension \(\nu_s = 2.05 \pm 0.09\). Such a low dimension chaos in a biological system as complex as the brain is striking. It shows the extreme coherence of the dynamical activity recorded by the
seizure by examining the divergence of two neighboring trajectories on the attractor. Let us consider the initial point V(t₀) on the phase space and another point on a close by trajectory. L(t₀) is the distance between these two points. The pair of points is allowed to evolve on their respective trajectories for time tₑ. Now the distance between the two trajectories is L(t₁), where t₁ = t₀ + tₑ. The largest Lyapunov exponent is given by

\[ \lambda = \frac{1}{tₑ} \ln \frac{L(t₁)}{L(t₀)}. \]

Another point in the neighborhood of V(t₀) is chosen and the procedure is repeated until all points in the time series are scanned. This procedure must converge to a constant value of the exponent \( \lambda \). The choice of the neighboring trajectory is not easy and is sensitive to the internal structure of the attractor; satisfactory results are found only in a narrow range of parameters. Moreover, all values of \( \lambda \) must converge toward a unique limit as \( tₑ \) is increased. Fig. 7 shows three trials out of a total of nine that were necessary to estimate the correct value of \( \lambda \). We find a positive value of the order of \( \lambda = 2.9 \pm 0.6 \). The inverse of this quantity gives the limit of predictability of the long-term behavior of the system. This time \( (=0.35 \text{ sec}) \) must be compared with the approximate pseudocycle of 0.35 sec of epileptic phenomena. Thus, there is a gradual loss of memory after each pseudocycle.

**Normal Brain Activity**

The EEG data recorded from the human brain during sleep cycles were also analyzed according to the procedure cited above (7). Chaotic attractors were identified for stage two and stage four of deep sleep. These attractors were characterized by a rather low dimensionality, which decreases as

**Fig. 8.** Phase portrait of EEG recorded from human sleep stage four following a procedure identical to that used in Fig. 2 (n = 2000 trials and \( r = 10^{-1} \)). The same \( \alpha = 270° \)
Table 1. Dimensionality of brain attractors

<table>
<thead>
<tr>
<th>Stages of brain dynamics</th>
<th>Embedding dimension $p$</th>
<th>Attractor dimension $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Awake</td>
<td>$p &gt; 9$</td>
<td>No saturation</td>
</tr>
<tr>
<td>Sleep 2</td>
<td>6</td>
<td>$5.03 \pm 0.07$</td>
</tr>
<tr>
<td>Sleep 4</td>
<td>5</td>
<td>$4.05 \pm 0.05$</td>
</tr>
<tr>
<td>REM</td>
<td>$p &gt; 9$</td>
<td>No saturation</td>
</tr>
<tr>
<td>Epilepsy</td>
<td>5</td>
<td>$2.05 \pm 0.01$</td>
</tr>
<tr>
<td>Lorenz</td>
<td>3</td>
<td>$2.01 \pm 0.01$</td>
</tr>
<tr>
<td>Rössler</td>
<td>3</td>
<td>$2.01 \pm 0.01$</td>
</tr>
</tbody>
</table>

Embedding dimension of the phase space and correlation dimension of the attractors for various stages of brain dynamics. Each value of $\nu$ corresponds to the EEG activity of a different subject. The Lorenz and Rössler attractors are shown in comparison. REM, rapid eye movement.

that chaotic attractors could be identified for several stages of normal and pathological brain activity indicates the presence of deterministic dynamics of a complex nature. This property should be related to the ability of the brain to generate and process information.

The low value of 2.05 for an episode of petit mal is striking, especially when it is contrasted with the values of 2.2–3.5 found for single neuron recordings from normal monkeys (15). In single neurons, the pseudocycles are of the order of $17–32$ msec, whereas in the case of a seizure, we are dealing with cooperative phenomena of the order of $300$ msec involving the entire cerebral cortex.

Unlike periodic phenomena, which are characterized by a limited number of frequencies, chaotic dynamics show a broad-band spectrum. Thus, chaotic dynamics increase the resonance capacity of the brain. In other words, although globally a chaotic attractor shows asymptotic stability, there is an internal instability reflected by the presence of positive Lyapunov exponents. This results in a great sensitivity to the initial conditions and, thus, an extremely rich response to external input.

In the light of such concepts, we may speculate further and suggest the following explanation for the type of petit mal epileptic seizure studied in this paper: the agents producing the seizure tend to drive the brain activity toward a stable periodic motion. In such states, information processing would be impossible and recovery would be extremely difficult. However, the brain manages to remain on a chaotic attractor, although one of a very low dimensionality, in order to process reflex activities.

The topological properties of the attractors and their quantification by means of dimensionality analysis may be an appropriate tool in the classification of brain activity and, thus, a possible diagnostic tool. For example, various forms of epileptic seizures could be classified according to their degree of coherence.

The determination of the minimum number of variables necessary for the description of epileptic attractors is a valuable clue for model construction. From our analysis of the epileptic attractor, we may suggest that at least five distinct variables are involved in the onset of petit mal. For example, two variables—the membrane potential of excitatory and inhibitory neurons—have the tendency to generate a periodic behavior, whereas three other variables pull back the attractor into a less coherent state. A model for epileptic seizure based on interaction of a group of excitatory and inhibitory neurons was shown to exhibit biphasic oscillations (16, 17). The model reported in ref. 17 based on interaction of one inhibitory and one excitatory cell was analyzed by using the range of parameters described in refs. 16 and 17. It was found that the differential delay equations describing the model show a stable homogeneous steady state for the delay $t' = 0.01$. However, for $t' = 0.1$, the periodic behavior sets in and is followed by a quasi-periodic behavior for $t' = 0.13$. For larger values of $t'$, the motion becomes chaotic.

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