Bursting oscillations from a homoclinic tangency in a time delay system

A. Destexhe \textsuperscript{1} and P. Gaspard
\[ C_m \frac{dX}{dt} = -g_L(X - V_L) - g_{EE}(X - V_E)F(X(t - \tau)) \]

\[ -g_{IE}(X - V_I)F(Y(t - \tau)) \]

fraction of excitatory or inhibitory cells active per unit of time.

Renormalizing the conductances by \( C_m \) leads to the following set of parameters (in ms\(^{-1}\)): \( \gamma = g_L / C_m \)
\( \Omega_1 = g_{EE} / C_m \)
\( \Omega_2 = g_{IE} / C_m \)
\( \Omega_3 = g_{EE} / C_m \) and \( \Omega_4 = \)
which appear from an infinite period bifurcation ($\Omega^*_f$ in fig. 1). As shown in fig. 1, a very similar diagram is seen for different values of the time delay.

Figure 2 shows the oscillating patterns seen at the approach of the point $\Omega^*_f$. For increasing values of $\Omega$, the limit cycle oscillations turn into bursting behavior.

We emphasize that the presence of the delay is necessary to observe this type of oscillations. Very similar behavior is observed for a very wide range of the parameters provided the delay is not too small, and the synaptic coefficients $\Omega_2$ and $\Omega_3$ are large enough. However, bursting oscillations disappear for time delays close to zero.

Figure 2 also shows that a bursting oscillation period is composed of an intermediate segment (the silent phase) and a burst of faster oscillatory behavior (the active phase). Very similar bursting oscillations have been observed in chemical systems [13].

A closer scrutiny of fig. 2 reveals that, during the approach to the critical point $\Omega^*_f$, the period continuously increases together with the number of pseudocycles of the fast oscillation, while the intermediate segment remains essentially unchanged. At the critical point $\Omega^*_f$, the oscillatory active phase lasts an infinite time, which leads us to assume the existence of a second limit cycle LC2 corresponding to the active phase fast oscillations. Since this limit cycle LC2 does not exist alone without the intermediate segment of the silent phase, we conclude that LC2 must be an unstable limit cycle of saddle type because trajectories like LC1 can enter the vicinity of LC2 before escaping from it. We now show that this behavior can be accounted for by the presence of a homoclinic tangency to the limit cycle LC2 as schematically illustrated in fig. 3.

As $\Omega_f$ increases, fig. 2 shows that the amplitude of the limit cycle LC1 increases with its period. Bursting oscillations appear when the limit cycle LC1 approaches the region of phase space where the unstable limit cycle LC2 exists (cf. fig. 3). A homoclinic tangency to LC2 can explain the properties of these oscillations. When the trajectory of LC1 is forced to pass in the vicinity of LC2, the limit cycle is distorted and fast oscillations appear transiently. In fig. 3a, a Poincaré section illustrates how the successive iterates of LC1 approach LC2. At the critical point, the trajectory of LC1 coalesces with LC2. The resulting homoclinic orbit is represented schematically.
Fig. 3. Schematic representation of a homoclinic tangency to an unstable limit cycle. The limit cycle (LC1) approaches progressively an unstable limit cycle (LC2) and tends to a homoclinic orbit (H) at the critical point when LC1 and LC2 are merging. A two-dimensional Poincaré section is schematized. This section is transverse to the limit cycle LC2 and includes the two slowest directions of the stable (S) and unstable (U) manifolds of LC2. (a) Close to the critical point, bursting oscillations are seen. The successive iterates of the trajectory of LC1 (black dots) approach LC2. (b) At the critical point, the limit cycle tends to a homoclinic orbit and the period is infinite. (c) Successive iterates represented in the XY plane, for each maximum of the variable X during one cycle of the oscillation of fig. 2d. The successive iterates approach and accumulate near LC2 before escaping (U).

Fig. 4. Logarithmic scaling of the period near the critical point. The period T is represented as a function of the parameter $\Omega_t$. The values $T_0 = 282.3 \pm 0.8$ ms, $\lambda = 0.1299 \pm 0.0014$ ms$^{-1}$ and $\Delta_0^p = 6.7186215 \pm 1.2 \times 10^{-7}$ were estimated from least squares fitting. Integration step $10^{-3}$ ms.

Fig. 5. Successive maxima of the variable $X$ during the end of the active phase. The maxima of $X$ are represented as a function of time for a value of $\Omega_t = 6.718617$ close to the critical point $\Omega_t^p$. The exponential least squares fitting of these points was realized and the following parameters were obtained: $X_{\max} = -7.85 \pm 0.63$ mV, $t_0 = 1100 \pm 2.6$ ms and $\lambda = 0.128 \pm 0.002$ ms$^{-1}$. Integration step $10^{-3}$ ms.

least squares fitting is $\lambda = 0.1299 \pm 0.0014$ ms$^{-1}$ (other parameters are given in the caption of fig. 4).

If we refer to the scheme of the Poincaré section of the system (fig. 3), then one can deduce that, close to the critical point, the successive iterates follow the unstable direction of the limit cycle (indicated by U in fig. 3c). These particular iterates correspond to the escape from the unstable limit cycle. The distance between each iterate and the unstable cycle should evolve exponentially, with an argument approaching $\lambda t$. Therefore, sufficiently close to the critical point, studying the successive iterates of LC1 on a Poincaré section should allow one to estimate the positive eigenvalue $\lambda$ of LC2. This value must be compared with that obtained from relation (2).

We realized a first return map of the system by considering the successive maxima $X_m(t)$ of the variable $X$ (fig. 5). This quantity seems to obey the following relation,

$$X_m(t) = X_{\max} - \exp[\lambda(t - t_0)]$$

where $X_{\max}$ and $t_0$ are constants. If we assume that $X_{\max}$ represents the amplitude of LC2, then the argument of the exponential constitutes an estimation of the eigenvalue corresponding to the unstable direction of LC2. The value obtained from least squares fitting (fig. 5) is $\lambda = 0.128 \pm 0.002$ ms$^{-1}$. A similar value is also obtained from the same procedure applied to the variable $Y$.

The value of $\lambda$ obtained from this first return map
of the system is remarkably close to that obtained from varied measurements using relation (9). The previously described mechanisms [6] shows that,