A COMPARATIVE STUDY OF THE EXPERIMENTAL QUANTIFICATION
OF DETERMINISTIC CHAOS

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Topological and correlation dimensions of physiological chaotic attractors are evaluated. The latter are embedded in phase spaces reconstructed using the lagging method, multi-channel recordings and the singular value decomposition technique. A comparative study shows that comparable results are obtained only if the correlation dimension is less than four.

In the last few years there has been an increasing enthusiasm for the physics of chaos. The realization that highly irregular and quasi-random behavior may be generated from rather simple deterministic dynamics led to the search for deterministic chaos in such diverse fields as hydrodynamics [1], chemistry [2], climatic variability [3,4] and brain dynamics [5–7]. These investigations were triggered by the work of Ruelle, Packard et al. [8] and Takens [9] which showed that a phase portrait topologically equivalent to that underlying a given dynamical system could be reconstructed from successive measurements of a single variable of the system.

Various algorithms have been proposed for obtaining from the experimental time series qualitative and quantitative information such as dimensions [10], entropies [11] or Lyapunov exponents [12]. One of the most popular ways of characterization of the chaotic dynamics is the evaluation of the fractal dimension of the corresponding attractors. Such a quantification provides a numerical criterion which makes possible a comparative study of various dynamical states of a given experimental system or interrelated systems. Once the dimension of the attractor is found, it is possible to estimate the minimum degrees of freedom necessary for describing the underlying dynamics.

There are many interrelated definitions of the fractal dimension of a chaotic attractor. Also several algorithms have been suggested for the evaluation of these dimensions from time series. Usually all these algorithms are tested with the attractors resulting from simple dynamical maps or low dimensional differential systems. They generally give good results for such simple systems with well defined dynamics, exhibiting a low fractal dimension. However the great challenge of chaotic dynamics is that it can be of some help in understanding real physical and physiological problems showing quasi-random behavior and which are evolving on attractors of relatively high dimension.

The algorithm of Grassberger and Procaccia [10] evaluates the correlation dimension $D_2$ of a chaotic attractor. The great advantage of $D_2$ is that it is readily computed from a time series and seems to give satisfactory results for high dimensional attractors. Recently, Broomhead and King [13,14] have introduced a method for the estimation of the topological dimension of a manifold from time series data which uses the technique of singular value decomposition.

In any dimension evaluation, the first step is the construction of a reliable phase space for embedding the attractor which represents the system's dynamics. Several procedures are available for such a construction. One may use the lagging method, simultaneous multi-channel recordings or the singular value decomposition procedure either using single or multi-channel recorded data.

In this paper, we have evaluated the correlation dimension as well as the topological dimension of
attractors constructed in several different ways and from the same time series representing brain and cardiac activity. The aim of this comparative study is to find the best strategy for a qualitative and quantitative study of the experimental data recorded from relatively high dimensional systems.

Phase portraits may be reconstructed from experimental time series in four different ways:

(i) Laging method [9]. In this procedure, one introduces a time lag τ, in principle arbitrary, in a single time series, thus creating an embedding space containing the attractor. When dealing with a finite set of data, not every value of the lag τ gives an acceptable embedding. Usually, only a narrow range of τ, which has to be found by comparative trials, provides phase portraits which can be used for dimension analysis.

(ii) Multi-channel trajectories. Eckmann and Ruelle [15] conjectured that the phase space spanned by simultaneous measurements of the same variable in different sites also constitutes an embedding. This construction has the advantage of covering the system spatially and thus incorporates better statistics. Although in this procedure one avoids the difficulties inherent to the laging procedure, another subjective quantity, namely the inter-site distance, appears in the problem. Moreover the latter has a lower bound fixed by the experimental recording technique.

(iii) Singular value decomposition. This procedure, used by Broomhead and King [13] provides a set of vectors for the construction of phase portraits. If \( \{v_1, v_2, ..., v_N\} \) are a succession of measurements of a given variable \( \nu \), then the method of delay is used to construct \( n \)-dimensional vectors \( \{x_1, x_2, ..., x_{N-n+1}\} \) where \( x_i^T = (v_i, v_{i-1}, ..., v_{i+n-1}) \). \( \{x_i^T\} \) are the rows of the rectangular trajectory matrix \( \mathbf{X} \). The product \( \mathbf{X}^T \mathbf{X} \) constitutes the covariance \( n \times n \) matrix of the system. The singular values \( \sigma_1, ..., \sigma_n \) are the square roots of the eigenvalues of \( \mathbf{X}^T \mathbf{X} \) and the corresponding eigenvectors constitute an orthonormal basis for the embedding space.

If the system is projected onto the more important axes, this procedure is a noise reducing technique (the finer twists of trajectories which are comparable in magnitude to the noise probably are also suppressed) and provides smoother phase portraits which however are very sensitive to the sampling interval of the data. Also an important step in the analysis is the choice of the window \( n - 1 \). Again the phase portrait is very sensitive to the choice of the window. As we shall see later, this choice is determinant for the evaluation of the topological dimension. Broomhead and King suggest that the window must be chosen such that the power spectrum contains no frequencies with significant magnitude greater than a cutoff frequency known as the band limit. In the case of complex and relatively high dimensional attractors, we find this procedure inapplicable. One is again led to finding the appropriate window by trial and error.

(iv) Singular decomposition of multi-channel data which combines the two above cited procedures.

All these four procedures have been used for the analysis of the time series obtained from the non-invasive measurements of the electrical activity of the brain. These so called electroencephalographic recordings (EEG) reflect the underlying activity of the brain tissue which involves the activity of some \( 10^{16} \) nerve cells. The etiology of the EEG is a function of the specific dynamical state of the brain. This activity switches between several states during the sleep cycles and in waking state, it shows two different characteristic behaviors whether the eyes are closed or open. In pathological situations, extremely regular and characteristic brain waves appear which are used as diagnostic tools. Figs. 1a-1e shows the EEG recordings of eyes closed (\( \alpha \) waves), deep sleep, Creutzfeld–Jakob (CJ) disease and human epileptic seizure. The last recording shows the electrical activity of a human heart (ECG). Figs. 1f-1l depicts the four phase portraits corresponding to CJ disease. Figs. 1f and 1g are respectively the portrait constructed with the help of laging method and multi-channel recordings. In figs. 1f and 1l the same single channel and multi-channel phase portraits were constructed using singular value decomposition.

The algorithm of Grassberger and Procaccia [10] for the evaluation of the correlation dimension is well known and will not be discussed here. Using the laging method, \( D_2 \) is computed from time series representing \( \alpha \) waves [16], deep sleep [5], CJ disease [17], human epilepsy [6] and also cardiac activity [18]. Correlation dimensions ranging between 2.03 and 6.2 have been found indicating the presence of
Fig. 1. Four second episodes of the human electroencephalogram and electrocardiogram (electrical activity): (a) eyes closed, (b) deep sleep, (c) epilepsy, (d) CI disease and (e) cardiac rhythm. Phase portraits of CI disease obtained from four different phase space constructions in three dimensions: (f) lagging method from one single measurement, (g) multi-channel construction from simultaneous measurements; (h) and (i) are respectively the (f) and (g) portraits constructed after singular value decomposition.

deterministic chaos in these important physiological states (see table 1).

When dealing with physiological data, the choice of an appropriate lag \( \tau \) is crucial. With an arbitrary \( \tau \) no saturation in \( D_2 \) versus the embedding dimension of the attractor is seen. Although for other values of \( \tau \) there will be saturation. Even when saturation exists, the value of \( D_2 \) is sensitive to the choice of \( \tau \). The data length also influences the value of \( D_2 \). As a rule, short data sets underestimate the value of the correlation dimension. One finds several suggestions in the literature regarding the choice of \( \tau \). The first zero of the autocorrelation function or the minimum of the mutual information \([19]\) does not always furnish satisfactory values for \( \tau \).

In our extensive experience with complex physiological data, we came to the conclusion that the best strategy is to take a \( \tau \) approximately equal to 25% of the pseudo cycle of the signal, evaluate \( D_2 \) and then repeat the calculation for a range of \( \tau \) around this value. In this way eventually a reasonably wide range of \( \tau \) is found where \( D_2 \) remains stationary if \( \tau \) is modified.

The choice of the best \( \tau \) is also related to the sam-

<table>
<thead>
<tr>
<th>System</th>
<th>Single channel correlation dimension</th>
<th>Multi-channel correlation dimension</th>
<th>Topological dimension</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>standard construction</td>
<td>singular vectors</td>
<td>standard construction</td>
</tr>
<tr>
<td>(a) cardiac rhythm (ECG)</td>
<td>3.2 ± 0.1</td>
<td>3.42 ± 0.01</td>
<td>2.9 ± 0.1</td>
</tr>
<tr>
<td>(b) CI EEG</td>
<td>3.7 ± 1.4</td>
<td>3.78 ± 0.05</td>
<td>3.8 ± 0.1</td>
</tr>
<tr>
<td>(c) alpha EEG</td>
<td>(1) 6.1 ± 0.5</td>
<td>6.2 ± 0.1</td>
<td>6.06 ± 0.05</td>
</tr>
<tr>
<td></td>
<td>(2) 6.3± 7.4</td>
<td>5.95 ± 0.05</td>
<td></td>
</tr>
<tr>
<td>(d) deep sleep EEG</td>
<td>4.4 ± 0.1</td>
<td>4.5 ± 0.1</td>
<td></td>
</tr>
<tr>
<td>(e) epileptic EEG</td>
<td>2.05 ± 0.09</td>
<td>2.03 ± 0.05</td>
<td></td>
</tr>
<tr>
<td>(f) Lorenz model</td>
<td>2.05± 2.1</td>
<td>2.05 ± 0.01</td>
<td></td>
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<tr>
<td>(g) white noise (for n=10)</td>
<td>9.4</td>
<td>9.4</td>
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Table 1. Correlation dimensions evaluated from different phase space constructions. The standard Grassberger-Procaccia algorithm is used and 1000 equidistant origins are taken from each calculation (the Theiler modification of the algorithm [20] brings out slight changes that fall within the range of errors). Data length \( N \), sampling frequency and range of \( \tau \) are respectively: (a) 60000, 250 Hz, 120-240 ms, (b) 60000, 250 Hz, 40-80 ms, (c1) 60000, 250 Hz, 12-40 ms, (c2) 18000, 1200 Hz, 25-46 ms, (d) 6000, 100 Hz, 100-200 ms, (e) 6000, 1200 Hz, 14-21 ms, (f) 20000, 50 Hz (integration step \( h = 0.02 \)), 80-160 ms, (g) 40000. Parameters for the Lorenz attractor: \( r = 28, P = 10, b = 8/3 \).
pling rate of the signal. For example in the case of alpha waves, $D_2$ remained constant for delay times of 5 to 10 ms and at a sampling frequency of 200 Hz.

In the case of alpha waves, CJ disease and cardiac activity, we could record the phenomena using a multiple channel setup. Therefore $D_2$ could be evaluated using both single and several simultaneous time series. If chaos is detected from a single channel, it indicates the presence of chaotic activity in the recorded site. Nothing guarantees that the time series from adjacent sites will indicate the same chaotic activity or if it did, we are dealing with the previous chaotic attractor. Therefore prior to the use of multi-channel analysis, each channel must be studied independently to make sure that all of them belong to the same dynamics. However here also one faces a new dilemma. How small must the inter-site distance be?

The results of multi-channel analysis are shown in table 1. For CJ disease, within the experimental error, $D_2 = 3.8 \pm 0.1$ is equal to the lowest value obtained with the laging method. The cardiac activity showed a value of $D_2 = 2.9 \pm 0.1$ in a relative good agreement with the value of $D_2 = 3.6 \pm 0.1$ obtained with the laging method from six channels. For all cases we have studied, usually the multi-channel $D_2$ has been found to be lower than the dimension calculated from each individual channel. For example, six leads of identical correlation dimension $(3.6 \pm 0.1)$ give a multi-channel dimension of $2.9 \pm 0.1$. This fact may be related to the observation that multi-channel phase portraits have a striking resemblance to the ones obtained from small lags $\tau$ and a small $\tau$ underestimates the value of $D_2$.

$D_2$ may also be evaluated with the help of vectors obtained from singular value decomposition of a single time series. From table 1, we see that for comparable windows, in general the laging method and the singular decomposition give comparable results. However with the latter technique usually one obtains rather extended scaling regions which make the $D_2$ estimation much more accurate. In the case of $\alpha$ waves, CJ disease and cardiac activity, the combination of singular value decomposition and multi-channel recordings gives similar results if compared with the simple multi-channel data.

The singular value decomposition has been extended to the evaluation of the local topological dimension [14]. A manifold $M$ is a topological space which locally is seen as $\mathbb{R}^m$. If $x_i \in M$, for a small enough neighborhood of $x_i$, the effect of curvature is unimportant and the manifold can be approximated by its tangent space at $x_i$. A local analysis around $x_i$ must be performed in a ball of radius $\epsilon$ centered on $x_i$. A neighborhood matrix $B_i(x_i)$ is constructed whose rows are $(x_i - x_k)^T$, $|x_j|$ are on the trajectory and $|x_i - x_k| \leq \epsilon$. It is shown that the rank of the matrix $B_i(x_i)$ is the dimension of the manifold $M$ when $\epsilon$ tends to zero.

In this analysis the singular vectors give a local coordinate system centered at $x_i$ whereas the local singular spectrum $\sigma_{i,A}(\epsilon)$, $\ldots$, $\sigma_{i,A}(\epsilon)$ indicates the disposition of data points inside the ball of size $\epsilon$ surrounding $x_i$. As $\epsilon$ increases, the eigenvalues corresponding to the tangent eigenvectors must increase linearly with $\epsilon$. An $\epsilon^2$ ($n \geq 1$) dependence shows the curvature of the manifold. The topological dimension is an integer and is a local quantity.

We have evaluated the topological dimension of the attractors reported in table 1. Special attention was paid to the case of CJ disease and cardiac activity as both attractors seem to exhibit two separate well defined regions in phase space.

In the context of low dimensional model systems, such as the Lorenz attractor, the local analysis seems to be independent of the choice of the window. However this does not seem to be the case for physiological time series. Two tangent directions are found for several choices of the window (fig. 2a).

In the case of CJ disease, if the window is chosen according to the band limiting frequency, a local dimension of two is found in different locations on the attractor. However if we limit the window only to 90% of the energy of the power spectrum, then the

Fig. 2. (a) and (b): scaling of the singular eigenvalues $\sigma_i(\epsilon)$ as a function of $\epsilon$ for the Lorenz attractor and alpha waves. The solid line is of slope unity.
topological dimension is two or three. The latter is in good agreement with the correlation dimension $D_2$.

In the case of the cardiac attractor, one sees two obviously different regions. The topological dimension computed from the QRS complex shows a value of two, whereas the P-wave exhibits a local value of three.

The alpha waves, which correspond to a high dimensional attractor, show a blurred scaling region (especially for $0.2 < \log \epsilon < 0.3$, not shown in fig. 2b) and it is difficult to assess the value of the topological dimension. At least two to four values show an $\epsilon$-dependence. However, one does not see a clear cut scaling (fig. 2b).

In order to get a global view of the attractor, the local topological dimension has to be evaluated in many different locations. The probability of the outcome of a given dimension should give more precise information about the geometrical structure of the attractor (research in progress).

The major interest of various algorithms which are developed for the evaluation of correlation and topological dimensions from experimental time series relies on the fact that they could be used for the entangling of complex dynamics encountered in physical and physiological systems. Table 1 shows that if the correlation dimension is less than four, all procedures considered in this paper furnish similar results indicating the relevance of existing algorithms for complex dynamics. The correlation dimension is a global quantity and in principle describes homogeneous attractors. However physiological attractors considered in this paper are most often inhomogeneous. Nevertheless we still feel that correlation dimension and topological dimension give a satisfactory view of the system's dynamics in the absence of other probes.

In the case of the cardiac attractor, the multi-channel $D_2$ is lower than the single channel dimension. The two different values of topological dimension show the inhomogeneity of the attractor. In the case of C1 disease, if only multi-channel recording is used, one could overlook the discrepancy of dimension seen in single channel evaluation. These variations may be the result of muscle or noise contamination or indicate the non-uniqueness of the attractor. In the case of alpha waves, the $D_2$ is rather high compared to the information furnished by the topological dimension.

However in high dimensional attractors the scaling regions are not unambiguous, especially for small values of $\epsilon$. This fact may reflect the sensitivity of the topological dimension to the length of the data set as well as to the sampling frequency. Some of our physiological data (alpha waves) consisted of 60000 points sampled at 200 Hz. It may be that such a data set still is not sufficient. In physiology, the longer experiments pose other problems such as for example the stationarity of the data set.

Because of the difficulties inherent to the choice of $\tau$, the lagging method requires relatively long computer time. Moreover the scaling regions are often very short, therefore a substantial error may be introduced in the evaluation of $D_2$. The evaluation of $D_2$ from a multi-channel phase portrait avoids these difficulties and provides quicker results. Here, one assesses the dynamics not only from a unique site but $D_2$ results from averaging of the system over a small area. Therefore the effect of accidental degeneracies or distortions, that could occur with a single observable, is minimized. In this method one must also check the stationarity of the results for several inter-electrode spacings. This method breaks down when the phase difference between the channels is very small.

Singular systems analysis provides good results when computing with a single or multi-channel recordings. In general, the scaling regions are longer and a slight decrease may be seen in the value of $D_2$. These facts could be a consequence of the noise reducing property (also the suppression of the finer twists of trajectories) of the singular decomposition. The most remarkable fact about this technique is that, in multi-channel recordings whenever the phase difference between channels is small, $D_2$ can be evaluated easily. The reason for this is that the information about the system's dynamics is no more scattered evenly among all channels but is concentrated on a few vectors. For example, we verified that a system formed from ten different linear combinations of the three-variable Lorenz model restitutes the three dynamical variables after singular decomposition. The correlation dimension remains of course unchanged in both cases. We must keep in mind that the singular value decomposition is window dependent and a search for the appropriate window is necessary.
If the necessary precautions in the handling of time series are taken, the dimension of chaotic attractors, together with other dynamical quantities such as Lyapunov exponents or Kolmogorov entropies, provides a good probe for the study of the dynamics of relatively high dimensional attractors underlying complex physiological processes. Especially if one considers that for such problems no other satisfactory dynamical approach is available and also what is important is the realization that these phenomena obey deterministic dynamics and that the relative values of attractors for the same system and not their absolute magnitudes are meaningful.

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